VIII.3.7 REVIEW QUESTIONS, EXAMPLES AND EXERCISES

REVIEW QUESTIONS

1. What is the main assumption in the method of ***separation of variables***?
2. What is a separation constant?
3. How does the Sturm-Liouville problem appear in the method of separation of variables?
4. What is the form of the solution for the initial value problem (IVBP) in the method of separation of variables?
5. How many terms are needed in the truncated infinite series for accurate representation of the solution?
6. Can you provide an example where the solution of the IBVP is described by **just a single-term** trigonometric function? How does this occur?

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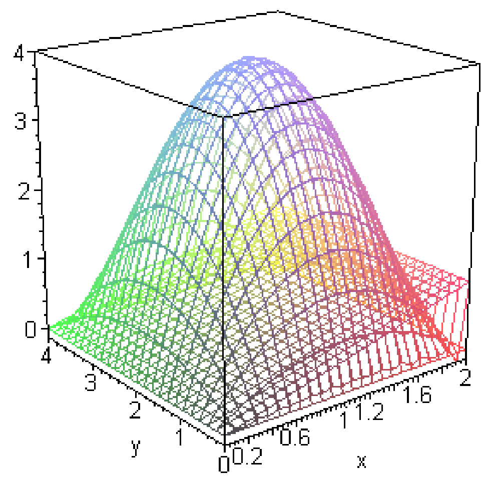
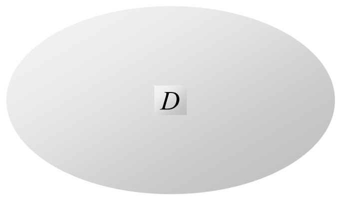


EXAMPLES AND EXERCISES 1. Let $D \subset \mathbb{R}^3$ be a ***domain*** (open connected set), and let $S = \overline{D} \setminus D$ be the ***boundary*** of D (recall Section VIII.1.11, p.568).

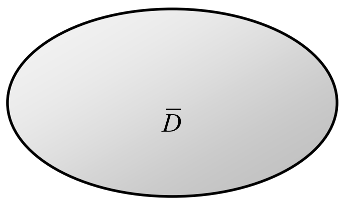
domain D is an open set



Show that if $\mathbf{r} \in S$ is a point on the boundary of D, then any open ball $B(\mathbf{r},R)$ with radius $R > 0$ contains points from both D and $\mathbb{R}^3 \setminus D$,
i.e. intersection of $B(\mathbf{r},R)$ with the domain and with the surroundings is not empty:

$$B(\mathbf{r},R) \cap D \neq \varnothing \text{ and } B(\mathbf{r},R) \cap \left(\mathbb{R}^3 \setminus \overline{D}\right) \neq \varnothing .$$

$\overline{D}$ is a closure of the domain D
$\overline{D}$ is a closed set



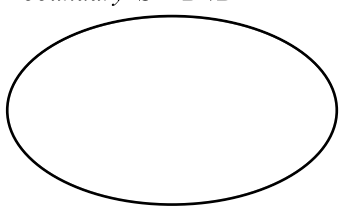
Remark: Boundary in a General Sense

This property is often used as a more general definition of the boundary.
If $A \subset \mathbb{R}^n$ is an arbitrary subset of $\mathbb{R}^n$ (not necessarily a domain) ,
Then a point $x \in \mathbb{R}^n$ is called *a* ***boundary point*** of A if,
for any radius $R > 0$:

$$B(x,R) \cap A \neq \varnothing \text{ and } B(x,R) \cap \left(\mathbb{R}^n \setminus A\right) \neq \varnothing .$$

The set of all such points $\partial A = \left\{ x \in \mathbb{R}^n,\ x \text{ is boundary point of } A \right\}$
is called the ***boundary*** of A in $\mathbb{R}^n$.

boundary $S = \overline{D} \setminus D$



If $S = \overline{D} \setminus D$ is the ***boundary*** of the domain D, then S also serves as the boundary of *S* in general sense too.

Examples of boundaries in the general sense:

a) $\partial(0,1] = \{0,1\}$

b) $\partial\{a\} = \{a\}$ (an insulated point is the boundary of itself)

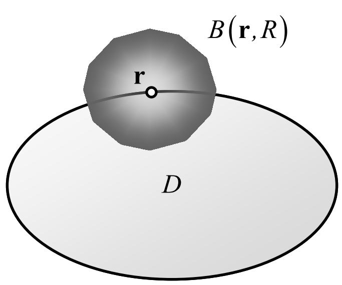
c) $\partial\ \mathbb{Q} = \mathbb{R}$

d) $\partial\ \mathbb{Z} = \mathbb{Z}$

e) $\partial\ \varnothing = \varnothing$

f) $\partial\ \mathbb{R}^n = \varnothing$

g) $\partial \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} \cup \{0\}$



2. a) Solve the Dirichlet problem for the Heat Equation:

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in [0,L],\ t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions:
$u(0,t) = 0$, $t > 0$ (Dirichlet)
$u(L,t) = 0$, $t > 0$ (Dirichlet)

b) Sketch the graph of solution for *L=3* and *a=0.1* and initial conditions:

i) $u_0(x) = 1$
ii) $u_0(x) = x(L - x)$
iii) $u_0(x) = \sin 2x$

3. The Superposition Principle for Non-Homogeneous Heat Equation with Non-Homogeneous Boundary Condition.:

Heat Equation:

$$\frac{\partial^2 u}{\partial x^2} + F(x) = a^2 \frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in (0,L),\ t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $u(0,t) = g_0,\ t > 0$ (Dirichlet)

$$\frac{\partial u(L,t)}{\partial x} = g_L, \quad t > 0 \qquad \text{(Neumann)}$$

Supplemental problems **a) steady state solution:**

$$\frac{\partial^2 u_s}{\partial x^2} + F(x) = 0 \qquad u_s(x): \quad x \in (0,L)$$

$$u_s(0) = g_0$$

$$\frac{\partial u_s}{\partial x}(L) = f_L$$

b) transient solution:

$$\frac{\partial^2 U}{\partial x^2} = a^2 \frac{\partial U}{\partial t} \qquad U(x,t): \quad x \in (0,L),\ t > 0$$

$$U(x,0) = u_0(x) - u_s(x)$$

$$U(0,t) = 0 \qquad t > 0$$

$$U(L,t) = 0 \qquad t > 0$$

First supplemental problem is a BVP for ODE.
The second supplemental problem is an IBVP problem for the homogeneous Heat Equation with homogeneous boundary conditions.

Show that $u(x,t) = U(x,t) + u_s(x)$ is a solution of the non-homogeneous IBVP.

Solve the problem with

$F(x) = 5,\ g_0 = 1,\ g_L = 3$ and $u_0(x) = x(4 - x)$.

Sketch the graph of the solution.

4. a) Solve the IBVP:

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} + F(x) \qquad u(x,t), \quad x \in (0,L), \; t > 0$$

initial condition: $u(x,0) = u_0(x)$

boundary conditions: $u(0,t) = f_1 \qquad t > 0 \qquad$ (I)

$$k \frac{\partial u(L,t)}{dx} + hu(L,t) = f_2 \qquad t > 0 \qquad \text{(III)}$$

b) Sketch the graph of solution with

$L = 4$, $a = 0.5$, $k = 2.0$,

$u_0(x) = x^2 - \frac{L}{2}x + 5$, $f_1 = 10$, $f_2 = 1$, $F(x) = x$

5. a) Solve the IBVP for the Heat Equation in the plane wall with distributed heat generation:

$$\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{1}{\alpha}\frac{\partial u}{\partial t}, \qquad u(x,t), \quad x \in (0,L), \quad t > 0, \qquad F(x) = \frac{\dot{q}}{k} x$$

Initial Condition: $u(x,0) = u_0(x)$

Boundary Conditions: $\frac{\partial u(0,t)}{\partial x} = 0 \qquad t > 0$

$$k \frac{\partial u(L,t)}{dx} = h_2 \left[T_\infty - u(L,t) \right] \qquad t > 0$$

b) Sketch the graph of solution with

$L = 0.5$, $\alpha = 0.0005$, $k = 150$,

$u_0(x) = 200$, $T_\infty = 10$, $h_2 = 250$, $\dot{q} = 200000$

6. a) Solve the Heat Equation in the cylindrical domain with angular symmetry

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = a^2 \frac{\partial u}{\partial t} \qquad u(r,z): \qquad 0 \le r < r_1, \; t > 0$$

Boundary condition: $u(r_1,t) = 0 \qquad t > 0$

Initial condition $u(r,0) = u_0(r)$

b) Sketch the graph of the solution for

$r_1 = 0.5$

$a = 3$

$u_0(r) = 6r^2 + 1$

7. a) Solve the Heat Equation in the cylindrical domain with angular symmetry

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=a^2\frac{\partial u}{\partial t} \qquad u(r,z), \qquad 0\le r<r_1, \qquad t>0$$

Boundary condition: $u(r_1,t)=f_1$ $\quad t>0$

Initial condition $u(r,0)=u_0(r)$

b) Display some creativity in visualization of solution for

$r_1=0.5$

$a=3000$

$f_1=70$

$u_0(r)=25r^2+20$

c) Give some physical interpretation of the problem

8. Solve the IBVP for the Heat Equation in polar coordinates with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=a^2\frac{\partial u}{\partial t} \qquad u(r,t), \quad r\in[0,r_1), \quad t>0$$

Initial conditions: $u(r,0)=u_0(r)$

Boundary condition: $k\dfrac{\partial u(r_1,t)}{\partial r}+hu(r_1,t)=f_1$ $\quad t>0$

And sketch the graph of solution for

$r_1=2$, $a=0.5$, $k=0.1$, $h=12$, $f_1=2$, and $u_0(r)=(r-r_1)^2$

(hint: first, find the steady state solution)

9. a) Solve the Heat Equation in the annular domain with angular symmetry (cylindrical wall with uniform heat generation)

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(r,t): \qquad r_1<r<r_2, \quad t>0$$

Boundary condition: $u(r_1,t)=T_1$ $\quad t>0$

$u(r_2,t)=T_2$ $\quad t>0$

Initial condition: $u(r,0)=u_0(r)$ $\quad r_1<r<r_2$

b) Display some creativity in visualization of the solution for

$r_1=0.5$	$T_1=50$
$r_2=0.6$	$T_2=10$
$k=150$	$u_0(r)=10$
$\alpha=0.00001$	$\dot{q}=500000$

10. EXAMPLE Radiation Induced Thermal Stratification in Surface Layers of Stagnant Water

Professor Raymond Viskanta (on the left)
Antalya, Turkey, June 2001

Radiation Induced Thermal Stratification in Surface Layers of Stagnant Water

Based on papers:

[1] D.M.Snider, R.Viskanta *Radiation Induced Thermal Stratification in Surface Layers of Stagnant Water,* ASME Journal of Heat Transfer, Feb 1975, pp.35-40.

[2] R.Viskanta, J.S.Toor *Radiant Energy Transfer in Waters,* Water Resources Research, Vol. 8, No.3, June 1972, pp. 595-608.

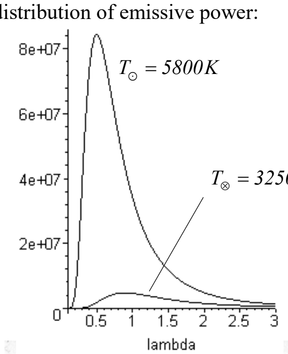
Introduction: The vertical temperature distribution in a body of water have important effects on chemical and physical properties, dissolved oxygen content, water quality, aquatic life and ecological balance as well as mixing processes in water.

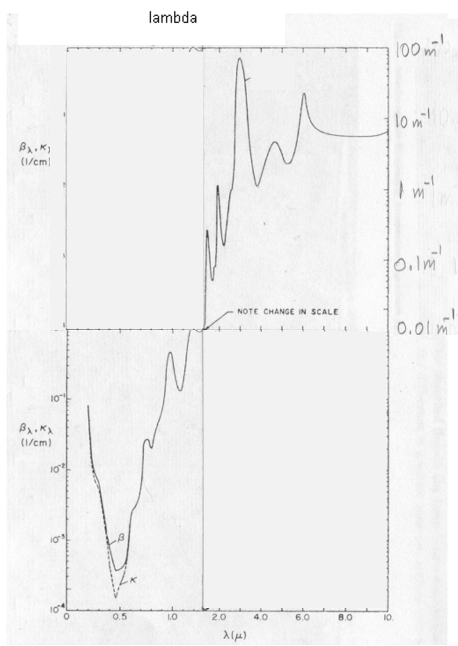
Solar radiation is recognized as the principle natural heat load in waters. Some investigators have considered the radiation to be absorbed at the water surface (i.e. opaque) and others treated the water as being semitransparent but ignored the spectral nature of radiation. Since the ultraviolet (UV) and infrared (IR) parts of the incoming solar radiation are largely absorbed within the first centimeters of the water and the visible part (VI) penetrates more deeply and carries significant energy to depths, the modeling of water as a gray medium is open to question and needs to be examined.

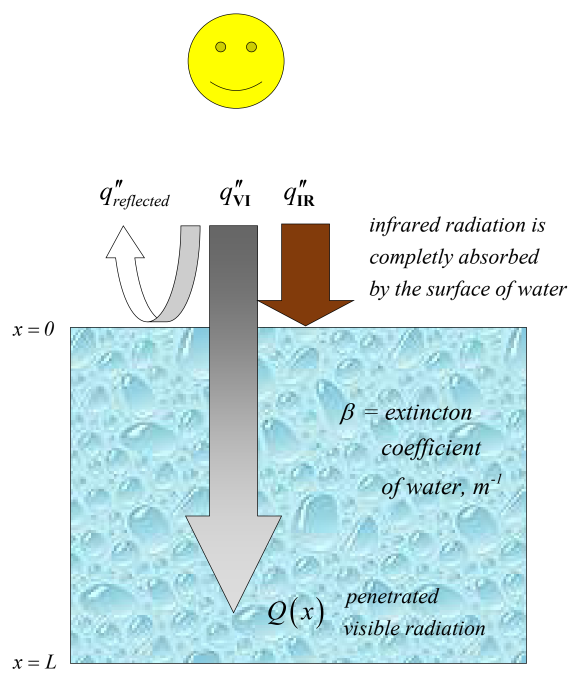
In the works of Raymond Viskanta (Purdue University) and coworkers, analysis for the time dependent thermal stratification of in surface layers of stagnant water by solar radiation was developed. The transient temperature distribution is obtained by solving the one-dimensional energy equation for combined conduction and radiation energy transfer using a ***finite difference method***. Experimentally, solar heating ($T_{\odot} = 5800K$) of water is simulated using tungsten filament lamps ($T_{\otimes} = 3250K$) in parabolic reflectors of known spectral characteristics.

Our Objective: ***Analytical*** investigation of transient combined conduction-radiation heat transfer with two band spectral model (VI-IR) of incident radiation.

Spectral distribution of emissive power:







$\beta_\lambda, \left[\frac{1}{cm}\right]$ *spectral absorption coefficient of liquid water*

Model:

Heat equation: $\frac{\partial^2 u}{\partial x^2} + \frac{Q(x)}{k} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$

Initial condition: $u(x,0) = u_0(x) = T_0$

Boundary conditions: $\left[-k\frac{\partial u}{dx} = -h_{eff}(u - T_\infty) + q''_{\mathbf{IR}}\right]_{x=0}$

$[u]_{x=L} = T_L$

Source function
(radiant energy absorption rate): $Q(x) = q''_{\mathbf{VI}}\beta e^{-\beta x}$

Water Properties:

Extinction coefficient	$\beta = 70$	m^{-1}
Density	$\rho = 1000$	$\frac{kg}{m^3}$
Specific heat	$c_p = 4180$	$\frac{J}{kg \cdot K}$
Conductivity	$k = 0.6$	$\frac{W}{m \cdot K}$

Data:

Length	$L = 0.381$	m
Temperature	$T_0 = T_{inf} = T_L = 25$	^oC
Visible irradiation	$q''_{\mathbf{VI}} = 850$	$\frac{W}{m^2}$
Infrared irradiation	$q''_{\mathbf{IR}} = 150$	$\frac{W}{m^2}$
Efficient convective coefficient	$h_{eff} = 12$	$\frac{W}{m^2 \cdot K}$

a. Solve the given IBVP:

$u(x,t) =$	
$u(0.05m, 3000s) =$ $28.3^o C$	*particular value*

b. Sketch the graph of the solution for $t = 5, 10, 15, 30, 60, 90\ min$ and compare with Viskanta's results.

c. Your view on the problem. How can the accuracy of the model be improved?
What have you learned from this problem?

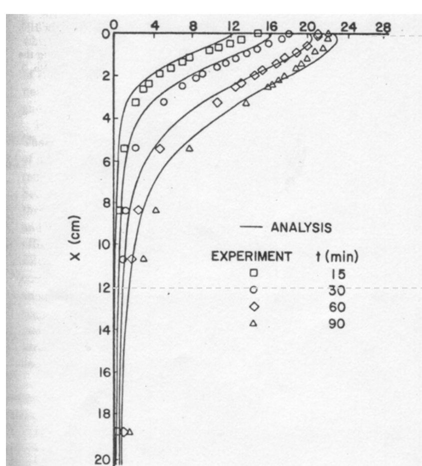


Fig. 4 Comparison of measured and predicted temperature distributions for lamps at 3250 K, $T_0 = T_\infty = 25$ C and $L = 38.1$ cm

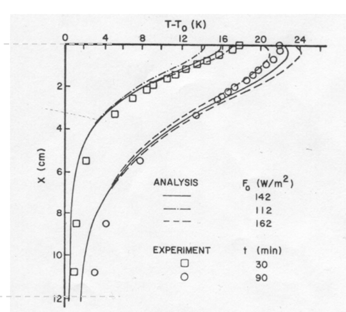


Fig. 6 Effect of variation in radiant energy flux considered absorbed at the surface on predicted temperature distribution for lamps at 3250 K, $T_0 = T_\infty = 25$ C, and $L = 38.1$ cm

Solution:

Heat equation: $$\frac{\partial^2 u}{\partial x^2}+\frac{Q(x)}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad Q(x)=q''_{\mathbf{VI}}\beta e^{-\beta x}$$

Initial condition: $$u(x,0)=u_0(x)=T_0$$

Boundary conditions: $$\left[-k\frac{\partial u}{dx}=-h_{eff}\left(u-T_\infty\right)+q''_{\mathbf{IR}}\right]_{x=0}=0$$

$$\left[u\right]_{x=L}=T_L$$

$$\left[-k\frac{\partial u}{dx}+h_{eff}u\right]_{x=0}=h_{eff}T_\infty+q''_{\mathbf{IR}}$$

$$\left[-\frac{\partial u}{dx}+\frac{h_{eff}}{k}u\right]_{x=0}=\frac{h_{eff}T_\infty+q''_{\mathbf{IR}}}{k}$$

$$\left[-\frac{\partial u}{dx}+Hu\right]_{x=0}=f_0 \qquad H=\frac{h_{eff}}{k},\ f_0=\frac{h_{eff}T_\infty+q''_{\mathbf{IR}}}{k}$$

$$\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad x\in(0,L) \qquad F(x)=\frac{q''_{\mathbf{VI}}\beta}{k}e^{-\beta x}$$

$$\left[-\frac{\partial u}{dx}+Hu\right]_{x=0}=f_0$$

$$\left[u\right]_{x=L}=T_L$$

$$u(x,0)=u_0(x)=T_0$$

I Steady State Solution:

$$\frac{\partial^2 u_s}{\partial x^2}+F(x)=0 \qquad x\in(0,L)$$

$$\left[-\frac{\partial u_s}{dx}+Hu_s\right]_{x=0}=f_0$$

$$\left[u_s\right]_{x=L}=T_L$$

$$\frac{\partial^2 u_s}{\partial x^2}+F(x)=0$$

$$\frac{\partial^2 u_s}{\partial x^2}=-F(x)=-\frac{q''_{\mathbf{VI}}\beta}{k}e^{-\beta x}$$

$$\frac{\partial^2 u_s}{\partial x^2}=-\frac{q''_{\mathbf{VI}}\beta}{k}\int e^{-\beta x}dx$$

$$\frac{\partial^2 u_s}{\partial x^2}=-\frac{q''_{\mathbf{VI}}\beta}{k}\frac{1}{(-\beta)}\int e^{-\beta x}d(-\beta x)$$

$$\frac{\partial u_s}{\partial x}=\frac{q''_{\mathbf{VI}}}{k}e^{-\beta x}+c_1$$

$$u_s=\frac{q''_{\mathbf{VI}}}{k}\int e^{-\beta x}dx+c_1x+c_2$$

$$u_s = -\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta x} + c_1 x + c_2$$

Boundary conditions:

$x = 0$

$$\left[-\left(\frac{q''_{\mathbf{VI}}}{k}e^{-\beta x} + c_1\right) + H\left(-\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta x} + c_1 x + c_2\right)\right]_{x=0} = f_0$$

$$-\left(\frac{q''_{\mathbf{VI}}}{k} + c_1\right) + H\left(-\frac{q''_{\mathbf{VI}}}{k\beta} + c_2\right) = f_0$$

$$-c_1 + Hc_2 = \frac{q''_{\mathbf{VI}}}{k}\left(1 + \frac{H}{\beta}\right) + f_0 \qquad f_0 = \frac{h_{eff}T_\infty + q''_{\mathbf{IR}}}{k}$$

$$-c_1 + Hc_2 = \frac{1}{k}\left[\left(1 + \frac{H}{\beta}\right)q''_{\mathbf{VI}} + h_{eff}T_\infty + q''_{\mathbf{IR}}\right]$$

$$-kc_1 + h_{eff}c_2 = \left(1 + \frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff}T_\infty$$

$x = L$

$$-\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L} + c_1 L + c_2 = T_L$$

$$c_1 L + c_2 = \frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L} + T_L$$

In matrix form:

$$\begin{bmatrix} -k & h_{eff} \\ L & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \left(1 + \frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff}T_\infty \\ \frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L} + T_L \end{bmatrix}$$

Use Cramer’s Rule:

$$det\begin{bmatrix} -k & h_{eff} \\ L & 1 \end{bmatrix} = -\left(k + h_{eff}L\right)$$

$$c_1 = \frac{det\begin{bmatrix} \left(1 + \frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff}T_\infty & h_{eff} \\ \frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L} + T_L & 1 \end{bmatrix}}{det\begin{bmatrix} -k & h_{eff} \\ L & 1 \end{bmatrix}} = \frac{\left(1 + \frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff}T_\infty - h_{eff}\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L} - h_{eff}T_L}{-\left(k + h_{eff}L\right)}$$

$$c_1 = \frac{\left(1 + \frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff}\left(T_\infty - T_L\right) - h_{eff}\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L}}{-\left(k + h_{eff}L\right)} \qquad c1 := \frac{h\ Tinf + qir + q0 + \frac{q0\ h}{k\ \beta} - \left(TL + \frac{q0\ \mathbf{e}^{(-\beta\ L)}}{k\ \beta}\right)h}{-k - L\ h}$$

$$c_2=\frac{det\begin{bmatrix}-k & \left(1+\frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}}+q''_{\mathbf{IR}}+h_{eff}T_\infty \\ L & \frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L}+T_L\end{bmatrix}}{det\begin{bmatrix}-k & h_{eff}\\ L & 1\end{bmatrix}}=\frac{-k\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L}-kT_L-L\left(1+\frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}}-q''_{\mathbf{IR}}L-h_{eff}T_\infty L}{-\left(k+h_{eff}L\right)}$$

$$c_2=\frac{\frac{q''_{\mathbf{VI}}}{\beta}e^{-\beta L}+kT_L+L\left(1+\frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}}+q''_{\mathbf{IR}}L+h_{eff}T_\infty L}{\left(k+h_{eff}L\right)}$$

$$c2:=\frac{-k\left(TL+\frac{q0\,\mathbf{e}^{(-\beta L)}}{k\,\beta}\right)-L\left(h\,Tinf+qir+q0+\frac{q0\,h}{k\,\beta}\right)}{-k-L\,h}$$

$$u_s=-\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta x}+\left[\frac{\left(1+\frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}}+q''_{\mathbf{IR}}+h_{eff}\left(T_\infty-T_L\right)-h_{eff}\frac{q''_{\mathbf{VI}}}{k\beta}e^{-\beta L}}{-\left(k+h_{eff}L\right)}\right]x+\frac{\frac{q''_{\mathbf{VI}}}{\beta}e^{-\beta L}+kT_L+L\left(1+\frac{h_{eff}}{k\beta}\right)q''_{\mathbf{VI}}+q''_{\mathbf{IR}}L+h_{eff}T_\infty L}{\left(k+h_{eff}L\right)}$$

II Transient Solution: $U(x,t)=u(x,t)-u_s(x)$

$$\frac{\partial^2U}{\partial x^2}=\frac{1}{\alpha}\frac{\partial U}{\partial t}\qquad x\in(0,L)$$

$$\left[-\frac{\partial U}{dx}+HU\right]_{x=0}=0\qquad \mathbf{R}$$

$$\left[U\right]_{x=L}=0\qquad \mathbf{D}$$

$$U(x,0)=u_0(x)-u_s(x)=U_0(x)$$

Supplemental SLP (RD):

λ_n roots of characteristic equation:

$X_n=sin\left[\lambda_n(x-L)\right]$ $\qquad \left\|X_n\right\|^2$

Solution: $U(x,t)=\sum_{n=1}a_nX_ne^{-\alpha\lambda_n^2t}$ $\qquad a_n=\frac{1}{\left\|X_n\right\|^2}\int_0^L\left[u_0(x)-u_s(x)\right]X_n(x)dx$

III Solution: $u(x,t)=u_s(x)+U(x,t)$

```
> U0:=subs(t=0,u(x,t)):
> U1:=subs(t=300,u(x,t)):
> U2:=subs(t=600,u(x,t)):
> U3:=subs(t=900,u(x,t)):
> U4:=subs(t=1800,u(x,t)):        30 min
> U5:=subs(t=3600,u(x,t)):
> U6:=subs(t=5400,u(x,t)):        90 min
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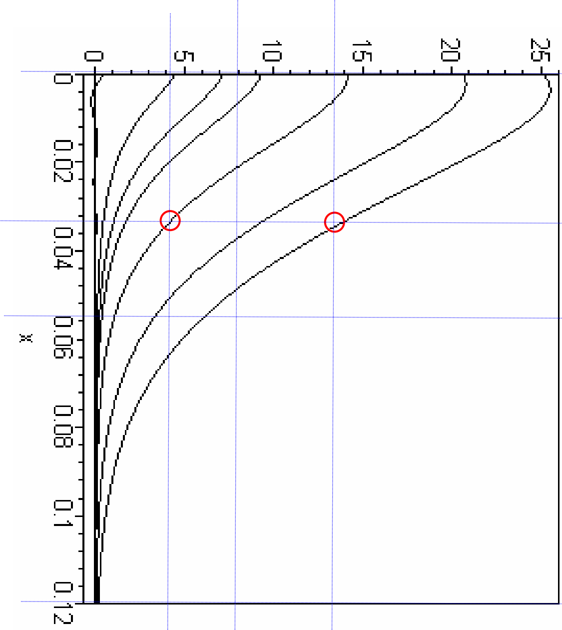
Comparison

Current analytical solution

Experimant and numerical solution [Viskanta]



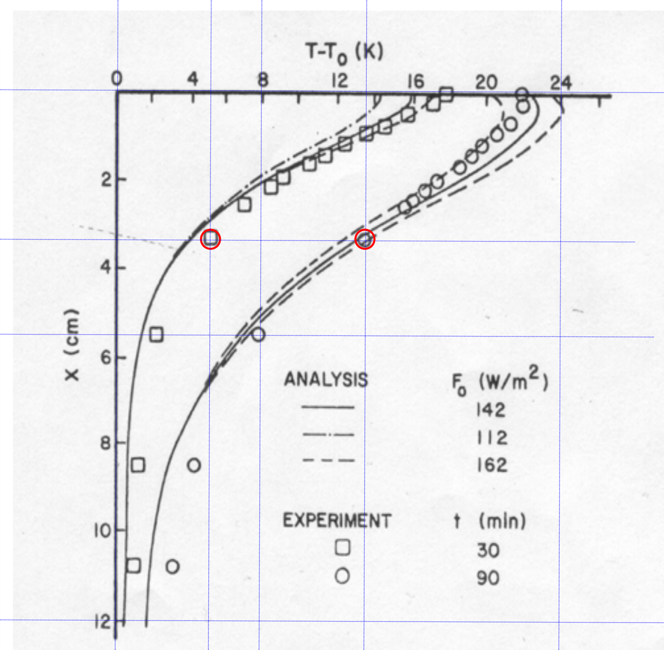
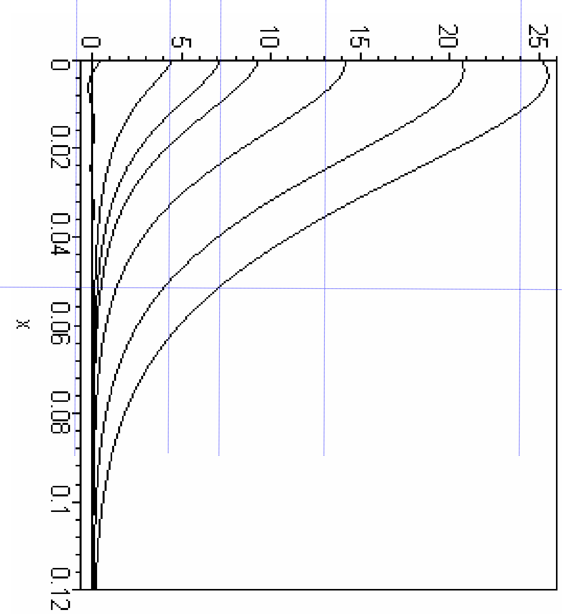


Fig. 6 Effect of variation in radiant energy flux considered absorbed at the surface on predicted temperature distribution for lamps at 3250 K, $T_0 = T_\infty = 25$ C, and $L = 38.1$ cm



11. Find the solution of the IBVP for the Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\ \ x \in (0,L),\ t > 0$$

initial condition: $u(x,0) = u_0(x)$

$\dfrac{\partial u(x,0)}{\partial t} = u_1(x)$

boundary conditions: $u(0,t) = 0 \qquad t > 0 \qquad$ (Dirichlet)

$u(L,t) = 0 \qquad t > 0 \qquad$ (Dirichlet)

Sketch the graph of solution with $L = 2$, $a = 0.5$, and

a) $u_1(x) = -0.1$, $u_0(x) = x^2(L-x)^2$

b) $u_1(x) = 0$, $u_0(x) = \sin\dfrac{6\pi}{L}x$

(observe the phenomena called standing waves)

12. Find the solution of the IBVP for the Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\ \ x \in (0,L),\ t > 0$$

initial condition: $u(x,0) = u_0(x)$

$\dfrac{\partial u(x,0)}{\partial t} = u_1(x)$

boundary conditions: $-u'(0,t) + H_1 u(0,t) = 0, \quad t > 0 \qquad$ (Robin)

$u(L,t) = 0 \qquad t > 0 \qquad$ (Dirichlet)

Sketch the graph of solution with $L = 5$, $a = 2.0$, and

a) $u_1(x) = 0.2$, $u_0(x) = (L-x)^2$

b) $u_1(x) = 0$, $u_0(x) = X_5(x)$ (eigenfunction)

13. a) Solve the IBVP:

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial u}{\partial t} + F(x) \qquad u(x,t),\ \ x \in (0,L),\ t > 0$$

initial condition: $u(x,0) = u_0(x)$

boundary conditions: $u(0,t) = f_1 \qquad t > 0 \qquad$ (Dirichlet)

$k\dfrac{\partial u(L,t)}{dx} + hu(L,t) = f_2 \quad t > 0 \qquad$ (Robin)

b) Sketch the graph of solution with

$L = 4$, $a = 0.5$, $k = 2.0$, $u_0(x) = x(x - L/2) + 5$, $f_1 = 10$, $f_2 = 1$, $F(x) = x$

14A. Find the solution for vibration of the annular membrane with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=a^2\frac{\partial^2 u}{\partial t^2} \qquad u(r,t),\quad r\in(r_1,r_2),\quad t>0$$

Initial conditions: $u(r,0)=u_0(r)$

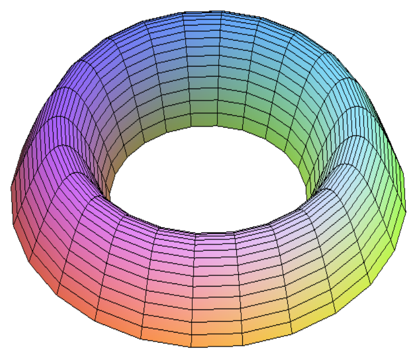
$\frac{\partial u}{\partial t}(r,0)=u_1(r)$

Boundary condition: $u(r_1,t)=0 \qquad t>0$

$u(r_2,t)=0 \qquad t>0$

And sketch the graph of solution for

$r_1=1,\ r_2=2\ \ a=0.5,\ u_0(r)=(r-r_1)(r_2-r),$ and $u_1(r)=0$.



14B. Heavy membrane

Find the solution for vibration of the annular membrane with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(r)=a^2\frac{\partial^2 u}{\partial t^2} \qquad u(r,t),\quad r\in(r_1,r_2),\quad t>0$$

Initial conditions: $u(r,0)=u_0(r)$

$\frac{\partial u}{\partial t}(r,0)=u_1(r)$

Boundary condition: $u(r_1,t)=0 \qquad t>0$

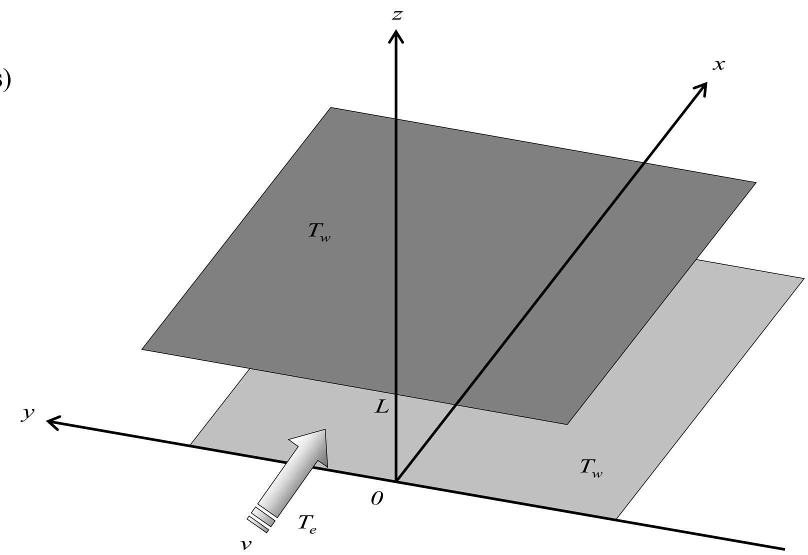
$u(r_2,t)=0 \qquad t>0$

And sketch the graph of solution for

$r_1=1,\ r_2=2\ \ a=0.5,\ F=-1.5,\ u_0(r)=(r-r_1)(r_2-r),$ and $u_1(r)=0$.

Non-Classical IBVPs

15. (**Flow Between Two Plates**)

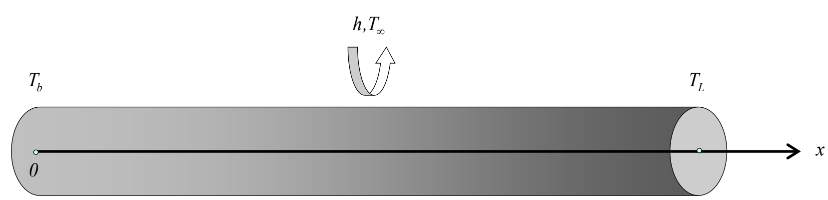


$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}-\frac{\rho c_p}{k}v\frac{\partial T}{\partial x}+\frac{\dot{q}}{k}=\frac{\rho c_p}{k}\frac{\partial T}{\partial t}$$

$x=0 \quad T=T_e$
$x\to\infty \quad T<\infty$
$z=0 \quad T=T_w$
$z=L \quad T=T_w$
$t=0 \quad T=T_0$

Find steady state solution for $\dot{q}=0$.
Sketch the graph for $T_e=80$, $T_w=10$, $v=2$, $L=0.02$, fluid is water.

16. (**Transient Conduction in the Fin**)



$$\frac{\partial^2 T}{\partial x^2}-\frac{hP}{kA_c}\left(T-T_\infty\right)+\frac{\dot{q}}{k}=\frac{\rho c_p}{k}\frac{\partial T}{\partial t}$$

$x=0 \quad T=T_b$
$x=L \quad T=T_L$
$t=0 \quad T=T_0$

Find transient state solution for $\dot{q}=0$.

circular copper fin ($D=0.005$)

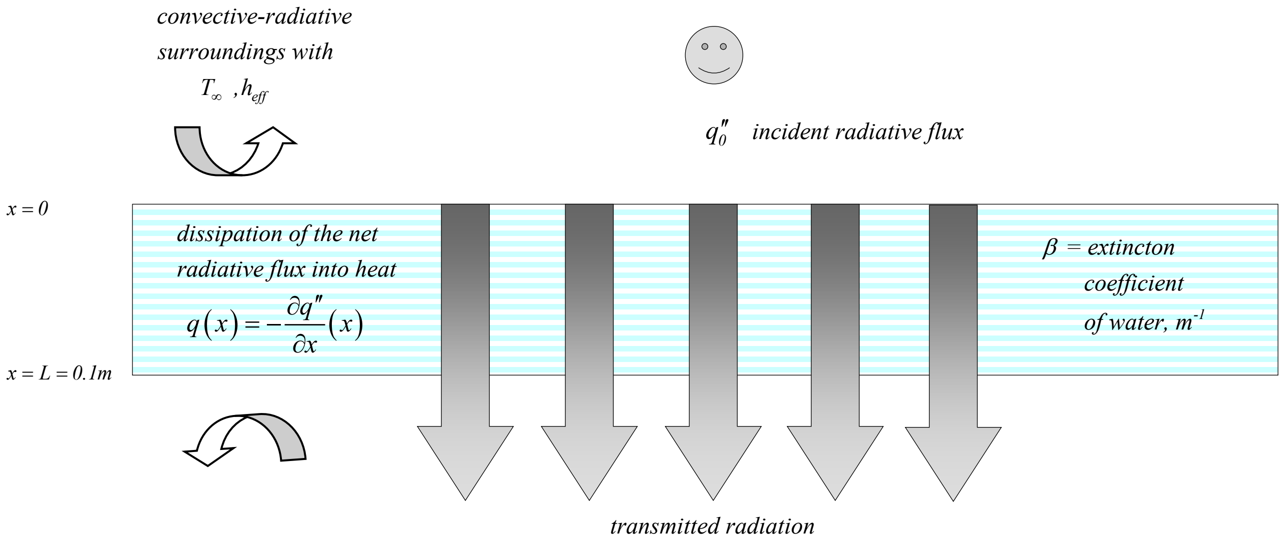
Sketch the graph for $T_b=200$, $T_L=50$, $T_\infty=10$, $T_0=10$, $h=150$, $L=0.2$,

17. [Based on Nellis&Klein, p.37] Absorption in a lens

Analytical investigation of transient combined conduction-radiation heat transfer with a gray spectral model of incident radiation.

A lens is used to focus the illumination radiation that is required to develop the resist in a lithographic manufacturing process

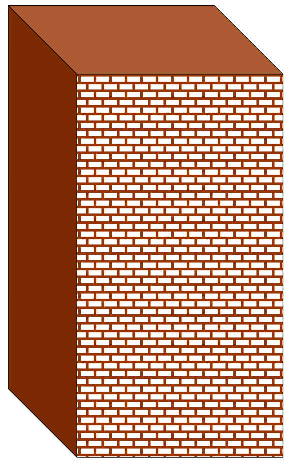
The lens is not perfectly transparent but rather absorbs some of the illumination energy that passes through it.



Model:	Heat equation:	$\frac{\partial^2 u}{\partial x^2} + \frac{Q(x)}{k} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$	
	Initial condition:	$u(x,0) = u_0(x) = T_0$	
	Boundary conditions:	$\left[-k\frac{\partial u}{dx} = -h_{eff}(u - T_\infty)\right]_{x=0}$	
		$\left[k\frac{\partial u}{dx} = -h_{eff}(u - T_\infty)\right]_{x=L}$	
	Dissipation source function (radiant energy absorption rate):	$Q(x) = q_0''\beta e^{-\beta x}$	
The Lens Properties:	Extinction coefficient	$\beta = 100$	m^{-1}
	Density	$\rho = 2500$	$\frac{kg}{m^3}$
	Specific heat	$c_p = 750$	$\frac{J}{kg \cdot K}$
	Conductivity	$k = 1.5$	$\frac{W}{m \cdot K}$
Data:	Length	$L = 0.1$	m
	Temperature	$T_{inf} = T_0 = 20$	^oC
	Incident radiative flux	$q_0'' = 1000$	$\frac{W}{m^2}$
	Efficient convective coefficient	$h_{eff} = 20$	$\frac{W}{m^2 \cdot K}$

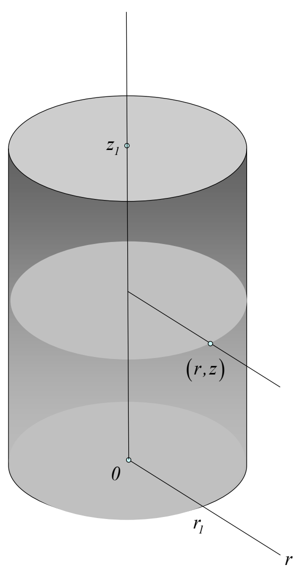
18. Investigate the temperature field in the long column of square cross-section two adjacent sides of which are thermally insulated and two others are maintained at temperatures $T_1 = 100^o C$ and $T_2 = 500^o C$ if initially it was of uniform temperature $T_0 = 20^o C$. Sketch the temperature surfaces.

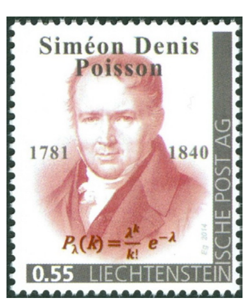
$L = 2m$

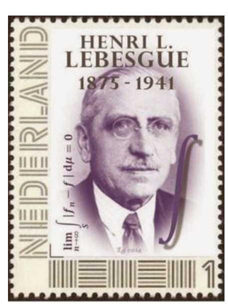


19. Use separation of variables for solution of IBVP for long cylinder with angular symmetry. $\left(\frac{\partial u}{\partial \theta} = 0\right)$:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$







20. Set up a mathematical model (choose an appropriate coordinate system and dimension of the problem, write the governing equation and corresponding initial and boundary conditions) for the following engineering models (do not solve the problem):

a) A very thin long wire dissipates energy in the massive layer of the stagnant media with the rate per unit length q, $\left[\frac{W}{m}\right]$. The media has a thermal conductivity k, $\left[\frac{W}{m \cdot K}\right]$. Determine the stationary temperature distribution in the media.

b) In the massive layer of homogeneous material (with thermal properties k, ρ, c_p) which was initially at the uniform temperature T_0, a localized heat source spontaneously started to dissipate energy with the rate q $[W]$. Determine the development of the temperature field in the material.

c) A very long tree trunk of radius R in the forest is exposed to the surrounding air (average wind speed is v $\left[\frac{m}{s}\right]$), but the dense crown prevents the direct sun radiation of the trunk. Set up the mathematical model describing the temperature distribution in the tree trunk during the day. Conductivity in the tree depends on direction: it is much higher along the tree than in the radial direction.

d) A wide reservoir of water of L meters deep is exposed to the solar irradiation G_0, $\left[\frac{W}{m^2}\right]$ incident at the angle θ. Penetration of the solar radiative flux along the path s is described by the Lambert-Beer Law $G(x) = G_0 \cos\theta \, e^{-\kappa s}$, where κ, $\left[\frac{1}{m}\right]$ is the gray absorption coefficient of water. Then the solar energy dissipated in water (radiative dissipation source or the divergence of radiative flux) is determined by $Q(s) = -\frac{dG(x)}{dx}$, $\left[\frac{W}{m^3}\right]$. Set up the mathematical model describing the equilibrium temperature field in the water layer.

e) Two opposite sides of the long column are insulated. There is an intensive condensation of the water steam on one of the other sides. The last side is exposed to the convective environment at temperature T_∞ and convective coefficient h, $\left[\frac{W}{m^2 \cdot K}\right]$. Due to some chemical reaction there is production energy in the column with the volumetric rate $\dot{q}$, $\left[\frac{W}{m^3}\right]$. Initially, column was at the uniform temperature T_0. Describe the transient temperature distribution inside of the column.

Stanislaw Mazur and Per Enflo

Stanislaw Mazur was a close collaborator of Banach at Lwów and a member of the Lwów School of Mathematics, where he actively participated in the mathematical discussions at the Scottish Café.

On 6 November 1936, he posed the "basis problem", which asked whether every Banach space has a Schauder basis, with Mazur promising a "live goose" as a reward. Thirty seven years later, Mazur fulfilled this promise, awarding a live goose to Per Enflo in a ceremony broadcast throughout Poland.

Lvov in 2009